Identity in Martin-Löf's type theory

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Martin-Löf's type theory. Introductory remarks

> *The theory of types with which we shall be concerned is intended to be a full scale system for formalizing intuitionistic mathematics as developed, for example, in the book by Bishop. (Martin-Löf 1973)*

Martin-Löf's type theory is a system of logic for constructive reasoning about mathematics.

Standard references are Martin-Löf *Intuitionistic Type Theory* (Naples 1984) and Nordström et al. *Programming in Martin-Löf Type Theory* (Oxford 1990).

The system has mainly interested constructive logicians and computer scientists.

More recently it has gained prominence as the 'type theory' of 'Homotopy Type Theory'.

Martin-Löf's type theory. More introductory remarks

Characteristic features:

- It is an interpreted formalism. Martin-Löf pays much attention to explaining the meaning of the primitive notions.
- The doctrine of types: any object belongs to a type. It is intrinsic to an object which type it belongs to.
- The Curry-Howard isomorphism: propositions are types. If A is a proposition, then the objects of type A are the proofs of A.
- Extension of the Gentzen-programme: the meaning of (most of) the primitive symbols is determined by their occurrence in formation, introduction, elimination, and equality rules.
  - The equality rules are rules of computation, or of reduction in the sense of Prawitz.
- A distinction is drawn between judgements and propositions.

Forms of categorical judgement

There are two basic forms of statement:

$a : A$	a is an object of type A.
$a = b : A$	a and b are the same objects of type A.

Since a proposition is a type, it is clear that these statements $a : A$ and $a = b : A$ cannot be propositions.

Martin-Löf has introduced the term 'judgement' (which is here not to be understood psychologically).

A proposition is true if it has a proof. Hence, if A is a proposition, $a : A$ says in effect that A is true. (We reserve the term 'proof' for such objects, or terms, a.)

The judgement $a : A$ may be compared with Frege's $\vdash A$, where A is a *judgeable content*.

Judgemental vs propositional identity

The logical operators $\wedge$, $\vee$, $\supset$, and $\neg$ operate on propositions, and $\forall$ and $\exists$ on propositional functions.

An identity statement $a = b : A$ is not a proposition, but a judgement, whence the logical operators are not applicable to it.

It is clear that in order to be able to express mathematics in the system, also identity propositions are needed.

Thus, we need a function **Id** which when applied to suitable arguments yields an identity proposition.

In particular, the function **Id** is to take a type A and two objects a and b of type A and yield a proposition $\mathbf{Id}(A, a, b)$.

Exploiting Currying, we think of $\mathbf{Id}(A)$ as a binary propositional function over A.

Judgemental identity: why is it needed?

An identity proposition $\mathbf{Id}(A, a, b)$ is formed by applying a propositional function $\mathbf{Id}(A)$ to suitable arguments a and b.

The propositional function $\mathbf{Id}(A)$ has a certain *domain* A.

We thus have the following order of conceptual priority:

$$\text{domain} \rightarrow \text{prop. function} \rightarrow \text{proposition}$$

But, to speak of a domain A, it must be clear what it is for elements of A to be the same elements of A.

For instance, it is implicit to our grasp of $\mathbb{Q}$ that $\frac{3}{4}$ and $\frac{6}{8}$ are the same $\mathbb{Q}$'s.

The identity constitutive of a domain A can therefore not be propositional.

Judgemental identity: why is it needed? Contd.

Cantor in his definition of *Mannigfaltigkeit* from 1882 requires that it should be determined whether

> *two objects belonging to the manifold are, in spite of formal differences in the mode of givenness, equal to one another, or not.*

Bishop makes a similar requirement in his definition of set.

In Martin-Löf's type theory to define a type A it must be laid down what it is for elements of A to be equal.

Laying down when elements of A are equal cannot be done by means of the propositional function $\mathbf{Id}(A)$.

It is done by means of identity judgements, $a = b : A$.

Judgemental identity. Some properties

Judgemental identity $a = b : A$ satisfies rules of reflexivity, symmetry, and transitivity; as well as rules of substitution.

Moreover, the equality rules associated with the primitive symbols are in effect non-trivial rules governing judgemental identity.

The equality rules say how certain functions are computed. We get equalities such as

$$\mathbf{fst}(\mathbf{pair}(a, b)) = a : A$$
$$\mathbf{ap}(\lambda(f), a) = f(a) : B$$

We may regard $a = b : A$ as saying that a and b are equal terms in a rich typed lambda-calculus.

Judgemental identity is sometimes called *definitional equality* and also *intensional equality*.

Propositional identity: formation rule

An identity proposition has the form $\mathbf{Id}(A, a, b)$.

The A cannot be any type. It must be what is called a *set* in Martin-Löf type theory. I shall write **set**.

A **set** is not a set in the sense of axiomatic set theory, but rather in the sense of an individual domain.

Thus we distinguish types of individuals and higher types, or function types. A **set** is a type of individuals.
(I shall not go into the details of the type hierarchy of MLTT here.)

If A is a **set**, $a : A$, and $b : A$, then $\mathbf{Id}(A, a, b)$ is a proposition.

$$\frac{A : \mathbf{set} \qquad a : A \qquad b : A}{\mathbf{Id}(A, a, b) : \mathbf{prop}} \quad (\mathbf{Id}\text{-form})$$

On introduction rules

A proposition A is a **set** the elements of which are the proofs of A.

We can assert that A is true if we have demonstrated $a : A$.

The conclusion of an introduction or elimination should therefore have the form $a : A$, rather than the simpler form 'A' as in Gentzen.

Hence in giving these rules we need to specify the form of the proof-term a in the conclusion $a : A$.

E.g., the introduction rule associated with $\wedge$ has the form

$$\frac{a : A \qquad b : A}{\mathbf{pair}(a, b) : A \wedge B}$$

In general, the form of the proof-term occurring in the conclusion of an introduction rule for a logical operator follows the BHK-interpretation.

Propositional identity: introduction rule

Gentzen and BHK cover only complex propositions.

Since $\mathbf{Id}(A, a, b)$ is atomic a new idea is needed in giving an introduction rule for $\mathbf{Id}$.

It is clear that $\mathbf{Id}(A, a, a)$ is true provided it is well-formed.

From the rules governing judgemental identity we can infer $c : \mathbf{Id}(A, a, b)$ from $c : \mathbf{Id}(A, a, a)$ and $a = b : A$.

Hence we decide on the following introduction rule:

$$\frac{a : A}{\mathbf{r}(A, a) : \mathbf{Id}(A, a, a)} \qquad (\mathbf{Id}\text{-intro})$$

Thus we stipulate that provided $a : A$ there is a proof $\mathbf{r}(A, a)$ of the proposition $\mathbf{Id}(A, a, a)$.

Intro- and elim-rules for inductively defined predicates I

Martin-Löf (1971) provided a general schema for introduction and elimination rules for inductively defined predicates.

The simplest form of introduction rule is

$$\frac{Qq(t) \qquad \dots \qquad Rr(t)}{Pp(t)}$$

E.g., we could introduce a predicate N by the two rules

$$N0 \qquad \frac{Nt}{Nst}$$

Thus, we simultaneously introduce the predicate N, the constant 0, and the function s.

We shall be more interested in a predicate E introduced by

$$Ett$$

Intro- and elim-rules for inductively defined predicates II

The elimination rule for a predicate P is to have the form

$$\frac{Pt \qquad \text{minor deductions}}{F(t)}$$

Associate with each of the predicates P, Q,..., R occurring in the introduction rule a new formula letter, $F(v)$, $G(v)$,..., $H(v)$ respectively.

Among the minor deductions there is to be for each introduction rule for P a deduction of the form

$$\begin{array}{ccc} G(q(x)) & \ldots & H(r(x)) \\ \ddots & & \iddots \\ & F(p(x)) & \end{array}$$

Intro- and elim-rules for inductively defined predicates III

For N this yields

$$\dfrac{Nt \qquad F(0) \qquad \begin{array}{c} F(x) \\ \vdots \\ F(sx) \end{array}}{F(t)}$$

For E introduced by the axiom Ett we get

$$\dfrac{Etu \qquad F(x,x)}{F(t,u)}$$

If we take $F(v,w)$ to be of the form $F'(v) \supset F'(w)$, the following rule is derivable.

$$\dfrac{Etu \qquad F(t)}{F(u)}$$

Hence, E is identity.

Propositional identity: elimination rule

The elimination rule for **Id** should be similar to that for E:

$$\frac{Etu \qquad F(x,x)}{F(t,u)}$$

The major premiss of **Id**-elimination will have to be $p : \mathbf{Id}(A, a, b)$.

The minor deduction for E is a proof of $F(x,x)$ with x free.

Thus we have a function of type

$$f : (x : A)F(x,x)$$

We get

$$\frac{p : \mathbf{Id}(A, a, b) \qquad f : (x : A)F(x,x)}{\mathbf{idel}(p, f) : F(a, b)} \qquad (\mathbf{Id}\text{-elim})$$

Note. The rule can be generalized by assuming the typing

$$F : (x : A)(y : A)(z : \mathbf{Id}(A, x, y))\mathbf{prop}$$

On equality rules

In the equality rules one can recognize reductions in the sense of Prawitz. Thus,

$$\dfrac{\dfrac{\begin{array}{c}\Pi \\ A\end{array} \qquad \begin{array}{c}\Pi' \\ B\end{array}}{A \wedge B}}{A} \quad \rightsquigarrow \quad \begin{array}{c}\Pi \\ A\end{array}$$

gives rise to

$$\mathbf{fst}(\mathbf{pair}(a, b)) = a : A$$

Likewise,

$$\dfrac{\dfrac{\begin{array}{c}A \\ \Pi \\ B\end{array}}{A \supset B} \qquad \begin{array}{c}\Pi' \\ A\end{array}}{B} \quad \rightsquigarrow \quad \begin{array}{c}\Pi' \\ A \\ \Pi \\ B\end{array}$$

gives rise to

$$\mathbf{ap}(\lambda(f), a) = f(a) : B$$

Propositional identity: equality rule

The conversion for the identity predicate E is

$$\dfrac{Ett \qquad \begin{array}{c}\Pi \\ F(x,x)\end{array}}{F(t,t)} \quad \rightsquigarrow \quad \begin{array}{c}\Pi[t/x] \\ F(t,t)\end{array}$$

Recall **Id**-introduction

$$\dfrac{a : A}{\mathbf{r}(A,a) : \mathbf{Id}(A,a,a)}$$

And **Id**-elimination

$$\dfrac{p : \mathbf{Id}(A,a,b) \qquad f : (x : A)F(x,x)}{\mathbf{idel}(p,f) : F(a,b)}$$

It is clear that **Id**-equality then must be

$$\mathbf{idel}(\mathbf{r}(A,a),f) = f(a) : F(a,a)$$

Judgemental vs propositional identity, again

The following rule is derivable:

$$\frac{a = b : A}{r(A, a) : \mathbf{Id}(A, a, b)}$$

In Martin-Löf (1984) **Id**-elimination is given as

$$\frac{p : \mathbf{Id}(A, a, b)}{a = b : A} \qquad \text{(Ext-id)}$$

This rule is difficult to justify, and accepting it makes type-checking undecidable.

(Ext-id) is *not* derivable from **Id**-elim. Without (Ext-id) there is thus in general no way of inferring $a = b : A$ from $p : \mathbf{Id}(A, a, b)$.

The two notions of identity thus differ in logical strength.

Frege's *Basic Law V*

Frege's Basic Law V is

$$\lambda(f) = \lambda(g) \iff \forall x(f(x) = g(x))$$

This is problematic when the terms on the LHS are taken to lie in the range of the quantifier on the RHS.

In MLTT the principle would be stated as

$$\mathbf{Id}(A \to B, \lambda(f), \lambda(g)) \iff (\forall x : A)\mathbf{Id}(B, f(x), g(x))$$

Martin-Löf remarks that even this formulation is problematic:

- The $\Rightarrow$-direction follows from **Id**-elimination;
- but, the $\Leftarrow$-direction is not justified.

Basic Law V, left to right

Let us first consider

$$\mathbf{Id}(A \to B, \lambda(f), \lambda(g)) \Rightarrow (\forall x : A)\mathbf{Id}(B, f(x), g(x))$$

Define a relation $R(v, w)$ on $A \to B$ by

$$R(v, w) \equiv (\forall x : A)\mathbf{Id}(B, \mathbf{ap}(v, x), \mathbf{ap}(w, x))$$

Then R is clearly a reflexive relation on $A \to B$.

Hence, if $\mathbf{Id}(A \to B, \lambda(f), \lambda(g))$ is true, then so is $R(\lambda(f), \lambda(g))$. By definition this is the same proposition as

$$(\forall x : A)\mathbf{Id}(B, \mathbf{ap}(\lambda(f), x), \mathbf{ap}(\lambda(g), x))$$

But this, in turn, is the same proposition as

$$(\forall x : A)\mathbf{Id}(B, f(x), g(x))$$

owing to the $\supset$-equality rule, $\mathbf{ap}(\lambda(f), y) = f(y) : B$.

Basic Law V, right to left

The converse

$$\mathbf{Id}(A \to B, \lambda(f), \lambda(g)) \Leftarrow (\forall x : A)\mathbf{Id}(B, f(x), g(x)) \qquad \text{(Vii)}$$

is, however, not justified by the **Id**-rules. (Appropriately formulated, this converse is called *function extensionality*.)

This can be shown model-theoretically. One can give a proof-theoretic argument along the following lines.

By normalization considerations one sees that in a derivation of (Vii) one would need to prove the higher-type judgemental identity $f = g : (A)B$ with only $(\forall x : A)\mathbf{Id}(B, f(x), g(x))$ as open assumption. But, that the latter holds provides no guarantee that f and g are equivalent terms. One counterexample is

$$f = [x : A \times B]x$$
$$g = [x : A \times B]\mathbf{pair}(\mathbf{fst}(x), \mathbf{snd}(x))$$

A valid form of *Basic Law V*

A form of Basic Law V that is justified in MLTT is obtained by replacing the propositional identities with suitable judgemental identities.

Now the propositional connective $\iff$ will not be applicable, so we instead get a pair of inferences:

$$\frac{f = g : A(B)}{\lambda(f) = \lambda(g) : A \to B} \qquad \frac{\lambda(f) = \lambda(g) : A \to B}{f = g : (A)B}$$

These are valid on grounds of the rules of $\supset$-intro, $\supset$-equality, and η-conversion.